Ball lightning as an example of a magnetohydrodynamic equilibrium

R. Kaiser[1] and D. Lortz[2]
[1]*Universität Bayreuth, D-8580 Bayreuth, Germany*
[2]*Max-Planck-Institut für Plasmaphysik, D-85748 Garching, Germany*
(Received 7 March 1995)

Magnetohydrodynamic equilibria appropriate for describing ball lightning are discussed in this paper. It is argued that lightning-induced fireballs should have a magnetic field decaying at infinity. Such equilibria, in fact, have a vanishing magnetic field outside a singly connected plasma region and are confined by the atmospheric pressure only. An equilibrium of this type with a spherical plasma boundary is investigated, and characteristic quantities are computed. Perturbation of the pressure and current profiles leads to deformation of the spherical plasma boundary, thus indicating the existence of a large class of equilibria with a nonspherical boundary. Finally, some remarks are made concerning the stability of these equilibria.

PACS number(s): 52.30.$-$q

I. INTRODUCTION

Ball lightning or fireballs are "luminous balls of some 10 cm in diameter, hovering or drifting through the air for seconds without contact with other bodies and vanishing either silently or with a bang [1]." Reports on observations of ball lightning have accumulated in such numbers (for a critical review see the books of Singer [2] and Barry [3], as well as the review article of Smirnov [4]) that its mere existence is hardly debated any longer, but its physical nature is debated all the more. For lack of a commonly accepted theory, it has been proposed [5] that the problem be split into three different (and possibly unconnected) partial problems, viz., the problem of confining gas or plasma stably for a few seconds (or even minutes) to a finite volume, the question of what mechanism is responsible for the observed electrical properties, and the problem of the energy source, which produces heat and light. Concerning the first partial problem, most answers are based on Hill's vortex in fluid dynamics [6] or on spheromaklike configurations in plasma physics [7]. Answers to the second problem are usually given on the assumption that the fireball is triggered by an ordinary stroke of lightning and the electrical properties derive from that origin. As far as the energy source is concerned, a major distinction is made between models that assume the energy to be stored (mostly in chemical form) inside the ball from the very beginning [4] and models that require an external source, e.g., high-frequency waves as suggested by Kapitza [8]. The present paper makes a contribution only to the first partial problem.

It is assumed that magnetohydrodynamic (MHD) is the appropriate setting to describe ball lightning and that axisymmetry is a good approximation to actual equilibria. [Only very few nonsymmetric equilibria are known and these exhibit special properties. It is, in fact, doubtful whether nonsymmetric equilibria with twisted magnetic lines exist at all.] Such equilibria are conveniently described as solutions of a free-boundary problem with the poloidal magnetic flux as a dynamic variable, which has to satisfy an elliptic equation containing pressure and poloidal current density as two free profile functions [9]. If linear profile functions are chosen, this equation separates and can be completely solved. A variety of equilibria emerge, all with constant or growing magnetic field at infinity, including equilibria analogous to Hills' vortex or the spheromak. These equilibria, however, are not suitable for describing fireballs for the following reason. The average total current observed in a stroke of lightning is of the order of 10^4-10^5 A [2]. If the lightning manages to move this current into a ball of 10 cm radius, then a magnetic field of the order of $0.1-1$ tesla is associated. In comparison with this, the geomagnetic field is negligible and cannot serve as a background field. This means that the lightning-induced magnetic field has to decay at infinity. From a simple energy argument, it then follows that the magnetic field has to vanish identically outside the singly connected plasma region. A family of such solutions is, in fact, already known: Prendergast used them to construct equilibria for a self-gravitating incompressible fluid sphere [10], they are implicitly contained in works on fusion-related equilibria by Morikawa, Rebhan, and Yeh [11], and Wu and Chen associated them with ball lightning [12]. These solutions are appropriate for a spherical volume and are labeled with the number m of magnetic axes contained in the sphere. If the magnetic field is assumed to be the maximal one, which can be confined in a sphere by an ambient pressure of 1 atm, the solutions are free of any parameters other than m and the radius R.

It is shown in this paper that for a sphere and linear profile functions, the solutions are in fact the only ones. The case $m=1$ is studied in greater detail; for $R=10$ cm, the magnitudes of the magnetic field and current are shown to agree well with the assumptions made above. Further quantities which, for example, are characteristic of the geometry or are important for stability considerations are derived. Perturbations of the linear profile functions are also considered. They are connected with deformations of the spherical boundary. It is shown that the perturbed equilibrium can be calculated to an arbitrary order, and for a quadratic perturbation of the pressure profile the first-order perturbation of the equilibrium is given explicitly. Finally, stability with respect to ideal MHD modes is considered. The equilibria with more than one magnetic axis turn out to be unstable with respect to axisymmetric perturbations, whereas the $m=1$ equilibrium is marginally stable in this regard. This simplest equilibrium, furthermore, satisfies all necessary criteria for stability, which supports the presumption that it is, in fact, stable. However, we have not yet been able to prove this.

The paper is organized as follows: In Sec. II the free-boundary problem is formulated, boundary conditions appropriate for ball lightning are derived, and solutions with spherical plasma boundary are presented. Characteristic numbers for these equilibria are computed and discussed in Sec. III. Section IV deals with perturbations of the equilibria and Sec. V contains some stability considerations. Section VI gives a summary. Some more technical details are collected in an Appendix. On the whole, we have attempted to write the paper in a way that is also accessible to MHD nonspecialists; therefore, some notions and equations well known to the experts are explained in greater detail than otherwise necessary.

II. EQUILIBRIA WITH SPHERICAL PLASMA BOUNDARY

The governing equations of ideal MHD equilibria are

$$\nabla p=\mathbf{j}\times\mathbf{B},\quad \mu_0\mathbf{j}=\nabla\times\mathbf{B},\quad \nabla\cdot\mathbf{B}=0\ , \tag{1}$$

where $\mathbf{B}$, $\mathbf{j}$, and p denote the magnetic field, current density, and pressure, respectively. In general, i.e., for fusion or astrophysics applications, the region Ω under consideration (which may be finite or all space) is subdivided into a highly conducting plasma region V bounded by a "magnetic surface" Γ, i.e., a surface with a vanishing normal component of the magnetic field and an insulating vacuum region $\hat{V}$, where pressure and current density vanish (see Fig. 1). Additionally, the normal component of the magnetic field is prescribed on $\partial\Omega$, or its asymptotic behavior is prescribed if Ω is all space. Γ is "free," i.e., it is not determined *a priori* but is part of the solution. For ball lightning it is appropriate to choose all space for Ω; the pressure in $\hat{V}$, however, does not vanish but takes a finite value corresponding to atmospheric pressure, and the magnetic field vanishes, in fact, identically in $\hat{V}$, as shown below.

Without further assumptions it is rather hard to find solutions of the above problem; no "general" solutions are known and it is not even clear whether the above problem is well posed or not [13]. This is the reason why axisymmetry is assumed in the following. Using cylindrical coordinates ρ, φ, z, one can write the general axisymmetric field in the form [9,14]

$$\mathbf{B}(\rho,\varphi,z)=\nabla\varphi\times\nabla\psi(\rho,z)+\mu_0 I(\psi)\nabla\varphi\ , \tag{2}$$

where ψ is a function of ρ and z only, and $I(\psi)$ is a free

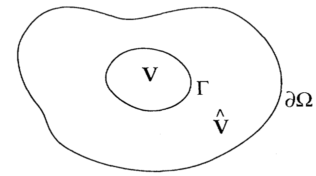

FIG. 1. Two-dimensional sketch of the plasma region V, vacuum region $\hat{V}$, free-boundary Γ, and outer wall $\partial\Omega$ (if existing).

profile function. $2\pi\psi$ denotes the poloidal magnetic flux through a ribbon in the equatorial plane, which is limited by the plasma boundary on one side and the surface $\psi=$const on the other, and $2\pi I$ denotes the poloidal current through that ribbon. With the ansatz (2) inserted in Eq. (1), the above problem reduces to the following elliptic problem for the flux function ψ [9]:

$$\Delta_*\psi+\mu_0^2 I'I+\mu_0\rho^2 p'=0\quad\text{in } V\ , \tag{3}$$

$$\Delta_*:=\frac{\partial^2}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{\partial^2}{\partial z^2}\ , \tag{4}$$

$$\Delta_*\hat{\psi}=0\quad\text{in } \hat{V}\ , \tag{5}$$

$$\psi=\hat{\psi}=0,\quad \nabla\psi=\nabla\hat{\psi}\quad\text{on } \Gamma\ . \tag{6}$$

Here, $p=p(\psi)\geq 0$ denotes another free profile function and prime means the derivative with respect to ψ. Note that on Γ not only ψ and $\hat{\psi}$ but also their derivatives have to match. If supplemented with a prescription of the asymptotic behavior of $\hat{\psi}$ (or boundary values on $\partial\Omega$ if Ω is finite) and with given profile functions $p(\psi)$ and $I(\psi)$, Eqs. (3)–(6) constitute a free-boundary problem for the unknowns ψ, $\hat{\psi}$, and Γ.

For nonlinear profile functions, the elliptic equation (3) is also nonlinear and still hard to solve. Explicit solutions are known only for linear profile functions

$$I=\lambda\psi/\mu_0\ , \tag{7}$$

$$p=p_0-\delta\psi/\mu_0\ , \tag{8}$$

with constants λ, p_0, δ, and this case will be concentrated on in this section. Furthermore, it is convenient to switch to spherical coordinates r, θ by

$$\rho=r\sin\theta,\quad z=r\cos\theta\ . \tag{9}$$

Then, Eq. (3) takes the form

$$\left[\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right]+\lambda^2\right]\psi=\delta r^2\sin^2\theta\ . \tag{10}$$

An inhomogeneous solution is

$$\psi_{\text{in}}=\begin{cases}(\delta/\lambda^2)r^2\sin^2\theta & \text{for } \lambda\neq 0\\ (\delta/10)r^4\sin^2\theta & \text{for } \lambda=0\end{cases} \tag{11}$$

and the homogeneous part of Eq. (10) separates with the ansatz

$$\psi_h=S(r)T(\cos\theta) \tag{12}$$

into

$$(1-t^2)\frac{d^2}{dt^2}T+cT=0 , \tag{13}$$

where the substitution $t:=\cos\theta$ has been used, and

$$\frac{d^2}{dr^2}S+\left[\lambda^2-\frac{c}{r^2}\right]S=0 . \tag{14}$$

Equation (13) has a countable set of regular eigensolutions $\{T_n\}$, $n\in\mathbb{N}_0$ with eigenvalues $c_n:=n(n-1)$. The T_n can be expressed by Legendre functions and, like these, are orthogonal (with respect to a suitable scalar product) and complete on the interval $-1\le t\le 1$ (see the Appendix). If c_n is inserted in Eq. (14), the solution regular in $r=0$ reads

$$S_n(r)=C_n\begin{cases}\lambda r j_{n-1}(\lambda r) & \text{for } \lambda\neq 0\\ r^n & \text{for } \lambda=0 ,\end{cases} \tag{15}$$

where $j_n(z)$ is a spherical Bessel function of the first kind [15], and the external solution reads

$$\hat{S}_n(r)=\hat{C}_n r^n+\hat{C}'_n r^{-n+1} . \tag{16}$$

Complete solutions of Eqs. (3) and (5) for the profile functions (7) and (8) are, therefore,

$$\psi(r,\theta)=\psi_{\text{in}}+\sum_{n\ge 2}S_n(r)T_n(\cos\theta) \tag{17}$$

and

$$\hat{\psi}(r,\theta)=\sum_{n\ge 2}\hat{S}_n(r)T_n(\cos\theta) . \tag{18}$$

Terms with $n\le 1$ have been omitted since they correspond to singular magnetic fields [see Eq. (2)]. In order to describe an equilibrium, ψ and $\hat{\psi}$ have still to meet the matching condition (6). Only a few equilibria are known and they all correspond to only the lowest term ($n=2$) in the expansions (17) and (18): [Note that $\psi_{\text{in}}\sim\sin^2\theta\sim T_2$ (see the Appendix).]

Hill's spherical vortex [6] has a purely poloidal magnetic field and is obtained for $\lambda=0$:

$$\psi=-\frac{\delta}{10}r^2(R^2-r^2)T_2,\quad \hat{\psi}=-\frac{\delta}{15}\frac{R^2}{r}(R^3-r^3)T_2 . \tag{19}$$

Here, R is the radius of the spherical plasma region.

The spheromak [7] is a force-free configuration, thus $\delta=0$:

$$\psi=C\lambda r j_1(\lambda r)T_2,\quad \hat{\psi}=-\frac{C}{3}\lambda^2 j'_1(\lambda R)\frac{1}{r}(R^3-r^3)T_2 . \tag{20}$$

The parameter λ is determined such that λR is a zero of j_1.

Finally, both solutions can be combined to describe an equilibrium with toroidal magnetic field and nonvanishing pressure gradient [11]:

$$\psi=\left[C\lambda r j_1(\lambda r)+\frac{\delta}{\lambda^2}r^2\right]T_2,\quad \hat{\psi}=\frac{A}{r}(R^3-r^3)T_2 . \tag{21}$$

Here, the amplitude A is given by

$$-3RA:=C\lambda[j_1(\lambda R)+\lambda R j'_1(\lambda R)]+2\frac{\delta}{\lambda^2}R \tag{22}$$

and λ is determined by the equation

$$C\lambda R j_1(\lambda R)+\frac{\delta}{\lambda^2}R^2=0 . \tag{23}$$

A common feature of these three solutions is the following: They describe a spherical plasma ball embedded in a background field, which is homogeneous at infinity. As pointed out in the Introduction, this is, however, not the right asymptotic behavior to describe fireballs. Their vacuum field should decay spatially at least as fast as a dipole field:

$$|r^3\mathbf{B}|\le\text{const} \quad \text{for } r\to\infty . \tag{24}$$

Furthermore, the normal component of the magnetic field B_n must vanish on the plasma boundary Γ. Such fields vanish, in fact, in the entire vacuum region $\hat{V}$. To realize this point, we introduce a potential Φ for $\mathbf{B}$,

$$\mathbf{B}=\nabla\Phi , \tag{25}$$

which satisfies

$$\nabla^2\Phi=0 \quad \text{in } \hat{V} , \tag{26}$$

as well as the boundary condition

$$\partial_n\Phi|_\Gamma=0 \tag{27}$$

and the growth condition corresponding to (24)

$$|\Phi|\le\text{const},\quad r^3|\nabla\Phi|\le\text{const} \quad \text{for } r\to\infty . \tag{28}$$

[As $\hat{V}$ is singly connected, Eq. (25) is globally valid.] If $\hat{V}$ is bounded, it is a well-known result from potential theory that the only solutions of Eqs. (26) and (27) are constant functions. If $\hat{V}$ is unbounded, as in our case, this is still true provided an additional growth condition such as (28) is satisfied. From Eqs. (26) and (27), it is concluded that

$$0=\int_{V_1}\Phi\nabla^2\Phi\, d^3\tau=-\int_{V_1}|\nabla\Phi|^2 d^3\tau+\int_{\Gamma_1}\Phi\partial_n\Phi\, d^2S , \tag{29}$$

where Γ_1 is a sphere with radius R_1, and V_1 is the volume between Γ and Γ_1. If R_1 tends to ∞, then because of the property (28) the last integral in Eq. (29) tends to zero, which means that there is only the trivial solution $\Phi=$const. Note that this result is valid for any singly connected plasma region and is independent of any symmetry assumption.

Such a solution with zero vacuum field is already known [10,12] and can easily be deduced from the third equilibrium given above. $\hat{\psi}$ in Eq. (21) vanishes if the amplitude A is zero. Equations (22) and (23) can then be considered as a homogeneous linear system of equations for the determination of the parameters C and δ. There are nontrivial solutions if the respective determinant vanishes:

$$\frac{1}{\lambda R} j_1(\lambda R) - j_1'(\lambda R) = j_2(\lambda R) = 0 \ . \tag{30}$$

Equation (30) is obviously satisfied if λR is one of the countably many zeros of j_2. From the linear system, we then have

$$\delta = -\lambda^4 \frac{j_1(\lambda R)}{\lambda R} C \tag{31}$$

and the fireball solution reads

$$\psi = CW(r)\sin^2\theta \ , \tag{32}$$

with

$$W(r) = \lambda r j_1(\lambda r) - \frac{j_1(\lambda R)}{\lambda R}(\lambda r)^2 \ . \tag{33}$$

For a spherical plasma volume, the solutions (32) and (33) are, in fact, the only ones. Another solution had necessarily to include higher terms in the general expansion (17). If the orthogonality of the T_n (see the Appendix) is used, the boundary condition (6) would result in the condition

$$j_{n-1}(\lambda R) = j_{n-1}'(\lambda R) = 0 \tag{34}$$

for all $n \geq 3$ with nonvanishing C_n. The j_n, however, have only simple zeros. Therefore, $C_n = 0$ for $n \geq 3$.

This section concludes with two remarks.

(1) It is well known that an argument based on the virial theorem

$$\int_V \left[3p + \frac{B^2}{2\mu_0} \right] d^3\tau = \int_{\partial V} \left[\left[p + \frac{B^2}{2\mu_0} \right] \mathbf{r}\cdot d\mathbf{S} - \frac{1}{\mu_0} \mathbf{r}\cdot\mathbf{BB}\cdot d\mathbf{S} \right] , \tag{35}$$

which excludes, for example, self-confined equilibria for fusion applications, does not work for ball lightning [1,12]. In the former case, the pressure in the plasma region is positive and the ambient pressure is zero. Sending ∂V to infinity and provided that the magnetic field is regular and decays sufficiently fast for $r \rightarrow \infty$, the right-hand side in Eq. (35) then vanishes, whereas the left-hand side is positive and a contradiction arises. For ball lightning, such a conclusion is prevented: A positive pressure in the plasma region is always balanced by a positive ambient pressure, and the right-hand side of Eq. (35) therefore does not vanish. For force-free equilibria, however, the argument works again since the pressure can be uniformly set to zero. This means that force-free equilibria are not appropriate for describing ball lightning, and this is true independently of any assumption concerning symmetry, the current profile, or the plasma boundary.

(2) Not yet solved is the question of whether for the linear profiles (7) and (8) the solution (32) and (33) is the only one, i.e., whether the sphere is the only possible free boundary. For a similar overdetermined free-boundary problem, viz.,

$$\nabla^2\psi = f(\psi) \ \text{in} \ V, \quad \psi|_{\partial V} = 0, \quad \partial_n \psi|_{\partial V} = \text{const} \ , \tag{36}$$

it has been proven that ∂V must be a sphere [16]; the big difference, however, is that the Laplacian–contrary to the Δ_*–is spherically symmetric and the proof in [16] is based on this fact. On the other hand, we will see in Sec. IV that deformations of the spherical plasma boundary are connected with perturbations of the linear profiles, which makes it likely that for fixed linear profiles the spherical plasma boundary is at least "locally" unique.

III. PROPERTIES OF THE FIREBALL EQUILIBRIUM

In this section some physical and geometrical properties of the fireball solution (32) and (33) are detailed. We concentrate on the simplest case, viz., λR is the lowest positive zero of j_2,

$$\lambda R =: z_1 \approx 5.76 \ . \tag{37}$$

All other cases turn out to be unstable with respect to ideal MHD modes (see Sec. V). The function $W(r)$ is plotted in Fig. 2; it has a single maximum at

$$r_0 \approx 0.513R \ . \tag{38}$$

The configuration has, therefore, a single magnetic axis situated at r_0 and $\theta_0 = \pi/2$. All other magnetic lines wrap in the poloidal and toroidal directions around this axis. They are confined to magnetic surfaces whose cross sections in the poloidal plane $\varphi = \text{const}$ are just the level lines of ψ (see Fig. 3).

The components of the magnetic field (2) read in spherical coordinates

$$B_r = -2C\frac{W(r)}{r^2}\cos\theta \ , \tag{39}$$

$$B_\theta = C\frac{W'(r)}{r}\sin\theta \ , \tag{40}$$

$$B_\varphi = C\lambda\frac{W(r)}{r}\sin\theta \ , \tag{41}$$

and the magnitude of the field is

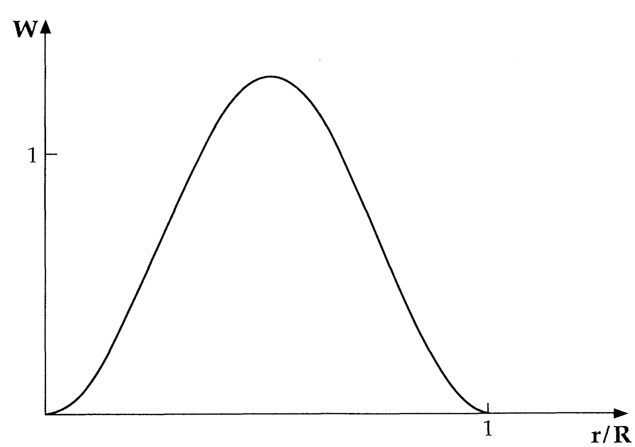

FIG. 2. Radial dependence of the dimensionless flux function ψ.

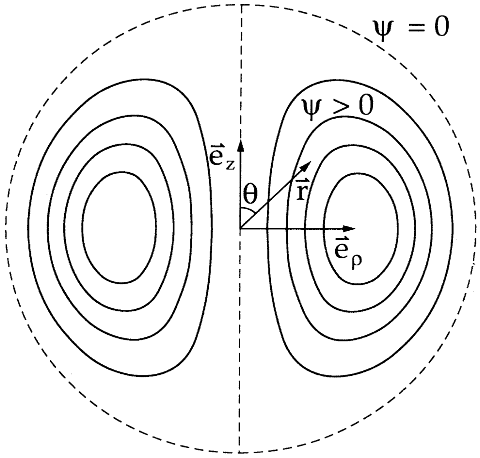

FIG. 3. Contour lines of ψ (dotted lines have $\psi=0$).

$$B^2=\frac{C^2}{r^2}[(W'^2+\lambda^2W^2)\sin^2\theta+\frac{4}{r^2}W^2\cos^2\theta] . \tag{42}$$

The level lines of B, which are plotted in Fig. 4, show that B has its maximum $B(0)$ at the center of the sphere. The ratio of B in the center and on axis is a characteristic quantity of the configuration and can be computed to give

$$\frac{B(0)}{B(r_0,\theta_0)}=2\left[\frac{1}{3}-\frac{j_1(z_1)}{z_1}\right]\frac{r_0}{R}\frac{z_1}{W(r_0)}\approx 1.65 . \tag{43}$$

Further characteristic quantities are the ratio of the current densities in the center and on axis or of the total poloidal current J_{pol} and the total toroidal current J_{tor}. As noted in Sec. II, the poloidal current through an equatorial ribbon limited by the plasma boundary and the surface $\psi=$const is given by

$$J_{pol}(\psi)=2\pi I(\psi) , \tag{44}$$

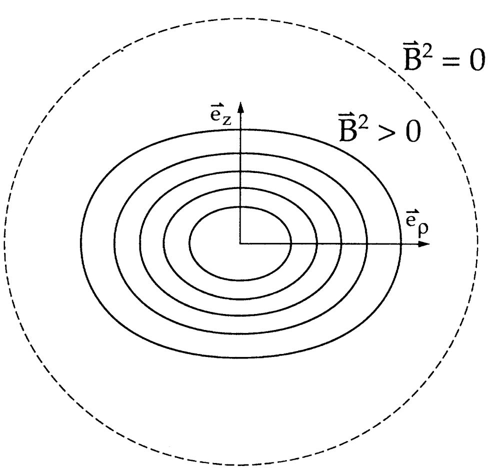

FIG. 4. Contour lines of the magnetic field strength B (dotted lines have $B=0$).

whereas the toroidal current density is given by [17, p. 110]

$$\begin{aligned} j_{tor}(r,\theta)&=-\frac{1}{\mu_0 r\sin\theta}\Delta_*\psi \\ &=r\sin\theta p'+\frac{\mu_0}{r\sin\theta}I'I(\psi(r,\theta)) \\ &=\frac{C}{\mu_0}\lambda^3 j_1(\lambda r)\sin\theta . \end{aligned} \tag{45}$$

Note that there is a reversal of the toroidal current density at $r_1\approx 0.780R$. Using these equations as well as (7), (8), and (31)–(33), we get for the ratio of the current densities in the center and on axis

$$\begin{aligned} \frac{j_{pol}(0)}{j_{tor}(r_0,\theta_0)}&=\frac{1}{j_{tor}(r_0,\theta_0)}\lim_{r\to 0}\frac{2\pi I(\psi(r,\pi/2))}{\pi r^2} \\ &=\frac{2\{1/3-[j_1(z_1)/z_1]\}}{j_1[(r_0/R)z_1]}\approx 2.04 . \end{aligned} \tag{46}$$

The current density is purely poloidal on the z axis and hence in the center and purely toroidal on the magnetic axis. Similarly, we get for the ratio of the total currents

$$\begin{aligned} \frac{J_{pol}}{J_{tor}}&=J_{pol}(\psi(r_0,\theta_0))/\int_0^R\int_0^\pi j_{tor} r\,dr\,d\theta \\ &=\frac{\pi W(r_0)}{\mathrm{Si}(z_1)-z_1 j_0(z_1)}\approx 2.10 , \end{aligned} \tag{47}$$

where Si denotes the sine integral $\mathrm{Si}(z)=\int_0^z[\sin(t)/t]dt$. The maximal magnetic field strength in the fireball depends on the ambient pressure p_0. If (31) and (32) are inserted in (8), it is found that the pressure decreases monotonically from p_0 at the plasma boundary to some value at the magnetic axis. The condition that the pressure does not become negative on axis limits the amplitude C to a maximal value C_{max},

$$C_{max}^2\lambda^4\left[-\frac{j_1(z_1)}{z_1}\right]W(r_0)=p_0\mu_0 . \tag{48}$$

For $p_0=1$ atm, the maximal field strength is

$$B_{max}(0)=2C_{max}\lambda^2\left[\frac{1}{3}-\frac{j_1(z_1)}{z_1}\right]\approx 1.33 \text{ tesla} , \tag{49}$$

which is in good agreement with the estimate in the Introduction. [Here, Système International units have been used with $\mu_0=4\pi\times 10^{-7}$ kg m/A^2 s^2.] Note that B_{max} is independent of the radius of the ball and well beyond the geomagnetic scale. The corresponding mean magnetic energy density can be computed to give

$$\begin{aligned} \varepsilon&=\frac{1}{(4/3)\pi R^3}\int_V\frac{B_{max}^2}{2\mu_0}d^3\tau \\ &=\frac{C_{max}^2\lambda^4}{\mu_0 z_1^3}\int_0^{z_1}\left[\left[1+\frac{1}{2z^2}\right]W^2(z)+W'^2(z)\right]dz \\ &\approx 7.6\times 10^4\frac{J}{m^3} \end{aligned} \tag{50}$$

[In the integral $r\lambda$ has been replaced by z. It can be solved analytically and has the value 5.40.] This means that for a ball radius of $R=0.1$ m the total magnetic energy is 320 J. Finally, the associated currents can be computed, e.g.,

$$J_{\text{pol}}=\frac{2\pi}{\mu_0 z_1}C_{\max}\lambda^2 W(r_0)\approx 2.09\times 10^6\ \text{R[m] A} . \quad (51)$$

For $R=0.1$ m this is about the current of a major stroke of lightning. Yet another characteristic quantity that is especially important for stability considerations is the so-called rotational transform ι. It measures the twist of the field lines around the magnetic axis and is defined by

$$\iota=\lim_{n\to\infty}\frac{1}{2\pi n}\sum_{\nu=1}^{n}\Delta\Theta_\nu , \quad (52)$$

where $\Delta\Theta_\nu$ is the poloidal angle made by the field line in the course of the νth toroidal transit. As field lines must not intersect, ι is the same for all field lines on a magnetic surface but may vary from surface to surface.

An alternative definition of ι or of its inverse q, the so-called MHD safety factor, uses poloidal and toroidal fluxes and is easier to evaluate for axisymmetric configurations [17, p. 112]:

$$q:=\frac{1}{\iota}=\frac{d\psi_{\text{tor}}}{d\psi_{\text{pol}}}=\frac{\lambda\psi}{2\pi}\oint\frac{dl}{r\sin\theta|\nabla\psi|} . \quad (53)$$

Here, the contour integral has to be done along the level lines of ψ in a poloidal section. The result, q as a function of ψ, is shown in Fig. 5. As for the spheromak [18], the safety factor decreases monotonically from the magnetic axis to the outside. The shear $S:=(\psi/q)(dq/d\psi)$, which measures the variation of q with ψ, is, however, about an order of magnitude smaller in the fireball than in the spheromak. Note that the limiting values of q can be calculated analytically. The value on the boundary $q(0)$ depends only on the twist along the z axis and is

$$q(0)=\frac{z_1}{2\pi}\approx 0.917 , \quad (54)$$

whereas the value on axis is determined by the half-axis ratio e of the elliptical cross section of ψ near the magnetic axis:

$$q(\psi_{\max})=\frac{z_1 r_0}{2eR}\approx 0.930 . \quad (55)$$

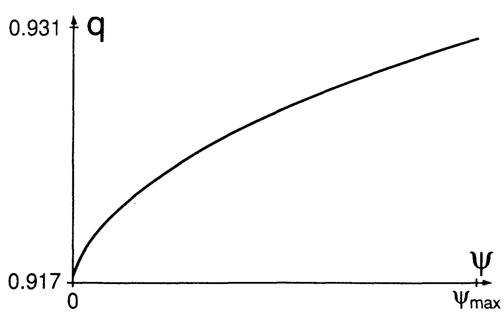

FIG. 5. The safety factor q vs the dimensionless flux function ψ.

[The numbers (54) and (55) have explicitly been reported to correct incorrect statements made by Wu and Chen [12].] The half-axis ratio e of the elliptical cross section in the neighborhood of the magnetic axis is another characteristic quantity of the equilibrium; the value used in (55) is

$$e=\left[\frac{-r_0^2 W''(r_0)}{2W(r_0)}\right]^{1/2}\approx 1.59 . \quad (56)$$

Remark: For comparison with the energy density value given in (50), one should mention the data available at present, as compiled in, for example, [3, p. 66]. The values derived from observations are quite large; they range from 0.4 J/cm^3 to 2.8×10^5 J/cm^3. Note, however, that these data are mostly based on rough estimates of the mechanical or thermal damage caused by the fireball and come from accidental observations. Measurements from various fireball experiments show typically lower energy densities. So, Barry gives a probable (and rather large) range of about 2×10^{-3}–2×10^2 J/cm^3 for the energy density.

As noted in the Introduction, the nature of the energy source for ball lightning is still controversial, and it depends on the answer to this problem whether the magnetic field makes a substantial (or the only) or only a negligible contribution to the total energy density. Therefore, the value (50) can only serve as a lower bound.

IV. PERTURBED FIREBALL EQUILIBRIA

Fireball equilibria as defined in Sec. II are solutions of the simplified free-boundary problem posed by Eq. (3) with given profile functions $I(\psi)$ and $p(\psi)$ and the boundary conditions

$$\psi=0 \quad \text{on } \Gamma , \quad (57)$$

$$\partial_r\psi=0 \quad \text{on } \Gamma . \quad (58)$$

[Equations (57) and (58) are equivalent to Eq. (57) and $\nabla\psi|_\Gamma=0$, as long as Γ is nowhere tangential to $\mathbf{r}$ (which we will assume).] To our knowledge there are, besides the solution with spherical boundary discussed in Sec. III, no further solutions known. Considering the example given in remark (2) of Sec. II, it is far from obvious whether further equilibria, especially with nonspherical boundary, exist at all. In this situation a perturbative study of the spherical equilibrium may be helpful. In fact, if perturbations of order ϵ are added to the profile functions (7) and (8), a solution of the problem (3), (57), and (58) is shown to exist at least in the sense of a formal power series in the parameter ϵ (the problem of convergence is left open). This solution corresponds to a deformed boundary, described by a function $\tilde{R}(\theta)$, which is likewise given as a power series in ϵ. Note that, if the perturbed equilibrium is required to be symmetric with respect to the equatorial plane (as the unperturbed equilibrium is), then the profile perturbations uniquely determine the perturbed equilibrium. Finally, for a quadratic perturbation of the pressure profile, the first-order perturbations of equilibrium and boundary are calculated explicitly.

In the following we assume the profile functions in the form

$$\mu_0^2 I' I(\psi)=\lambda^2\psi-\epsilon u(\psi)\ , \tag{59}$$

$$-\mu_0 p'(\psi)=\delta+\epsilon v(\psi)\ , \tag{60}$$

with analytic functions u and v:

$$u(\psi)=\sum_{i=2}^{\infty} u_i\psi^i,\quad v(\psi)=\sum_{i=1}^{\infty} v_i\psi^i\ , \tag{61}$$

and for ψ and $\tilde{R}$ a power series ansatz in ϵ is made,

$$\psi=\sum_{i=0}^{\infty}\psi_i\epsilon^i\ , \tag{62}$$

$$\tilde{R}=\sum_{i=0}^{\infty}\tilde{R}_i\epsilon^i\ , \tag{63}$$

where ψ_0 is the unperturbed equilibrium and $\tilde{R}_0=R$ is constant. Let us now consider the Nth order in ϵ of Eq. (3) and Eqs. (57) and (58). Equation (3) gives

$$(\Delta_*+\lambda^2)\psi_N=P_1(\psi_0,\ldots,\psi_{N-1})r^2\sin^2\theta + P_2(\psi_0,\ldots,\psi_{N-1})\ , \tag{64}$$

where P_1 and P_2 are power series in the variables $\psi_0,\ldots,\psi_{N-1}$. Equations (57) and (58) are more complicated to evaluate. If (63) is inserted in (62) and use is made of the Taylor series expansion of ψ_i with respect to r,

$$\psi_i(\hat{R},\theta)=\sum_{j=0}^{\infty}\frac{1}{j!}\frac{\partial^j}{\partial r^j}\psi_i(R,\theta)(\hat{R}-R)^j\ , \tag{65}$$

the Nth-order contribution of Eq. (57) is of the form

$$0=\sum_{i,j,\sigma_1,\ldots,\sigma_N=0}^{N} C_{i,\sigma_1,\ldots,\sigma_N}\frac{\partial^j}{\partial r^j}\psi_i\tilde{R}_1^{\sigma_1}\cdots\tilde{R}_N^{\sigma_N} =\psi_N+\sum_{i,j,\sigma_1,\ldots,\sigma_{N-1}=0,i\neq N}^{N} C_{i,\sigma_1,\ldots,\sigma_{N-1},0}\times\frac{\partial^j}{\partial r^j}\psi_i\tilde{R}_1^{\sigma_1}\cdots\tilde{R}_{N-1}^{\sigma_{N-1}}\ . \tag{66}$$

Here, $C_{i,\sigma_1,\ldots,\sigma_N}$ are constants, ψ_i is taken at radius R but still depends—like $\tilde{R}_i$—on the variable θ, and the indices i, j, $\sigma_1,\ldots,\sigma_N$ are subject to the restrictions

$$\sum_{l=1}^{N}\sigma_l=j,\quad \sum_{l=1}^{N} l\sigma_l+i=N\ . \tag{67}$$

Note that the expression (66) contains no derivatives of ψ_N, and $\tilde{R}_N$ does not occur at all. The latter point is due to the fact that $\tilde{R}_N$ is multiplied by $(\partial/\partial r)\psi_0(R,\theta)$, which is zero. For the Nth-order contribution of Eq. (58) we get similarly

$$0=\sum_{i,j,\sigma_1,\ldots,\sigma_N=0}^{N} C_{i,\sigma_1,\ldots,\sigma_N}\frac{\partial^{j+1}}{\partial r^{j+1}}\psi_i\tilde{R}_1^{\sigma_1}\cdots\tilde{R}_N^{\sigma_N} =\tilde{R}_N\frac{\partial^2}{\partial r^2}\psi_0+\sum_{i,j,\sigma_1,\ldots,\sigma_{N-1}=0}^{N} C_{i,\sigma_1,\ldots,\sigma_{N-1},0}\times\frac{\partial^{j+1}}{\partial r^{j+1}}\psi_i\tilde{R}_1^{\sigma_1}\cdots\tilde{R}_{N-1}^{\sigma_{N-1}}\ . \tag{68}$$

[The restrictions (67) apply, of course, to Eq. (68), too.] Now, if $\psi_0,\ldots,\psi_{N-1}$ and $\tilde{R}_1,\ldots,\tilde{R}_{N-1}$ are known, Eqs. (64) and (66) constitute a well-posed boundary value problem for ψ_N. The solution is unique since the only solution ψ_h of the corresponding homogeneous problem,

$$\psi_h\sim r\lambda j_2(\lambda r)T_3(\cos\theta)\ , \tag{69}$$

is antisymmetric with respect to the equatorial plane (see the Appendix), which contradicts our initial symmetry assumption. If ψ_N is known, the boundary perturbation $\tilde{R}_N$ is determined by the algebraic equation (68). Note that ψ_0 contains the factor $\sin^2\theta$, which vanishes at $\theta=0$, π. But from Eq. (64) it follows recursively that all $\psi_i, i\in\mathbb{N}$ behave as

$$\psi_i\sim O(\sin^2\theta) \tag{70}$$

in the limit $\theta\to 0,\pi$. Therefore, the function $\tilde{R}_N(\theta)$ is well defined on the interval $0\le\theta\le\pi$. This shows that the coefficients in the power series (62) and (63) can be determined to any order.

The simplest perturbation of the linear profile functions is given by

$$u\equiv 0,\quad v(\psi)=v_1\psi \tag{71}$$

in Eqs. (59) and (60). Equation (64) then reads for $N=1$

$$(\Delta_*+\lambda^2)\psi_1=v_1\psi_0 r^2\sin^2\theta=v_1 r^2 W(r)\sin^4\theta\ , \tag{72}$$

where the constant C in ψ_0 has been set to 1, and Eqs. (66) and (68) take the form

$$\psi_1(R,\theta)=0\ , \tag{73}$$

$$\tilde{R}_1(\theta)\frac{\partial^2}{\partial r^2}\psi_0(R,\theta)+\frac{\partial}{\partial r}\psi_1(R,\theta)=0\ . \tag{74}$$

With the symmetric ansatz

$$\psi_1(r,\theta)=f(r)\sin^2\theta+g(r)\sin^4\theta\ , \tag{75}$$

Eq. (72) reduces to

$$f''+\left[\lambda^2-\frac{2}{r^2}\right]f=\frac{8}{r^2}g\ , \tag{76}$$

$$g''+\left[\lambda^2-\frac{12}{r^2}\right]g=v_1 r^2 W\ , \tag{77}$$

where primes denotes the derivative with respect to r, and from Eqs. (73) and (74) we have

$$f(R)=g(R)=0\ , \tag{78}$$

$$\tilde{R}_1(\theta) = -\frac{1}{W''(R)}[f'(R)+g'(R)\sin^2\theta] . \quad (79)$$

As the homogeneous part of Eq. (77) admits only one solution, which is regular in $r=0$, the complete solution of Eq. (77) depends on a single parameter a:

$$g(r) = -v_1 r^4 \left[\tfrac{1}{6}j_0(\lambda r) + \frac{j_1(\lambda R)}{\lambda R}\right] + a\lambda r j_3(\lambda r) . \quad (80)$$

With (80) inserted in the right-hand side of Eq. (76), the complete solution of Eq. (76) can be calculated:

$$f(r) = \frac{v_1}{\lambda^2} r^2 \left[\tfrac{1}{3}\lambda r y_0(\lambda r) + 8\frac{j_1(\lambda R)}{\lambda R}\right] - 4aj_2(\lambda r) + b\lambda r j_1(\lambda r) , \quad (81)$$

where y_0 is a spherical Bessel function of the second kind and the last term is the only homogeneous solution of Eq. (76). The parameters a and b are determined by Eq. (78), where the fact is used that both $j_1(\lambda R)$ and $j_3(\lambda R)$ are nonzero. Note that $f'(R)$ and $g'(R)$ are nonzero also. They determine in Eq. (79) the first-order deformation of the boundary, which may be prolate or oblate depending on the sign of v_1.

It remains for future work to prove rigorously the existence of nonspherical equilibria and explore thoroughly what boundary shapes are possible.

V. STABILITY CONSIDERATIONS

In this section some aspects of linear stability in ideal MHD are considered. Since the equilibria under consideration are axisymmetric, the general linear perturbation can be expanded in modes $\sim\exp(in\varphi)$ and modes with different toroidal wave numbers n can be investigated separately. Here, only the cases $n=0$ (axisymmetric modes) and $n\to\infty$ ("ballooning" modes) are considered more closely. Special axisymmetric modes allow us to discard all equilibria with more than one magnetic axis ($m\geq 2$) as unstable, whereas the lowest equilibrium ($m=1$) is marginally stable with respect to these modes. Ballooning modes are known to furnish severe stability criteria and often set the stability threshold in equilibria of fusion interest. The $m=1$ equilibrium turns out to be ballooning stable, too. Finally, rigid motions are discussed, which, for example, are responsible for the instability of the spheromak.

In order to prove instability for $m\geq 2$, a test function is constructed which makes the MHD energy functional (see, for instance, [17, p. 251]) negative. For axisymmetric perturbations, the functional can be minimized analytically with respect to the components of the displacement vector ξ that are tangential to the magnetic surface $\psi=$const. For the case where no surface currents are present, the result is a functional in the (in general complex) variable $X=\xi\cdot\nabla\psi$ in the plasma region V and in the variable $\hat{X}$ in the vacuum region $\hat{V}$ (up to the factor ρ, $\hat{X}$ is the toroidal component of the vector potential of the magnetic field perturbation in $\hat{V}$) [19]:

$$W = W_V + W_{\hat{V}} , \quad (82)$$

$$W_V = \tfrac{1}{2}\int_V \frac{d^3\tau}{\rho^2}\{|\nabla X|^2 + f^2 + \gamma p\rho^2 g^2 - [\mu_0^2(I'I)' + \mu_0\rho^2 p'']|X|^2\} - \tfrac{1}{2}\int_\Gamma \frac{j_{\text{tor}}|X|^2}{\rho|\nabla\psi|} d^2S , \quad (83)$$

$$W_{\hat{V}} = \tfrac{1}{2}\int_{\hat{V}} \frac{d^3\tau}{\rho^2}\{|\nabla\hat{X}|^2 + \gamma p\rho^2\hat{g}^2\} . \quad (84)$$

Here, γ is the ratio of the specific heats, j_{tor} is given in Eq. (45), and the following abbreviations have been used:

$$f = I\left\langle \frac{X}{\rho^2}\right\rangle' \Big/ \left\langle \frac{1}{\rho^2}\right\rangle , \quad (85)$$

$$g = \langle X\rangle'/\langle 1\rangle, \quad \hat{g} = \langle \hat{X}\rangle'/\langle 1\rangle . \quad (86)$$

The angles

$$\langle \cdots \rangle = \oint \cdots \frac{\rho}{|\nabla\psi|} dl \quad (87)$$

denote the average over a magnetic surface, which in the case of axisymmetry is just the line integral along a curve $\psi=$const., $\varphi=$const. The second contribution in Eq. (84) is due to the nonvanishing ambient pressure in fire ball equilibria. [Averaging in $\hat{V}$ is performed with respect to an arbitrary continuation $\hat{\psi}$ of ψ into $\hat{V}$.] The boundary conditions are

$$X = \hat{X} \quad \text{on } \Gamma \quad (88)$$

and

$$\hat{X} = 0 \quad \text{on } \partial\Omega \quad (89)$$

or

$$\int_{\hat{V}} |\nabla\hat{X}|^2 \frac{d^3\tau}{\rho^2} < \infty \quad (90)$$

if $\partial\Omega$ does not exist. In addition, X must vanish where $\nabla\psi$ vanishes.

For the case where there is more than one magnetic axis ($m\geq 2$), we have

$$\lambda R \geq z_2 \approx 9.095 , \quad (91)$$

where z_2 is the second zero of $j_2(z)$. Consider a test function X, which is antisymmetric with respect to the equatorial plane in V and which is zero in $\hat{V}$, $\hat{X}\equiv 0$. Then f and g vanish and for the linear profiles (7) and (8) the energy functional (82) reduces to

$$W = W_V = \tfrac{1}{2}\int_V \frac{d^3\tau}{\rho^2}\{|\nabla X|^2 - \lambda^2|X|^2\} . \quad (92)$$

Minimizing (92) under the boundary conditions

$$X|_\Gamma = X|_{\rho=0} = X|_{z=0} = 0 \quad (93)$$

leads to the eigenvalue problem

$$\Delta_* X+\kappa^2 X=0\ ,$$
$$X(R,\theta)=X\left[r,\frac{\pi}{2}\right]$$
$$=X(r,0)=X(r,\pi)=0 \qquad (94)$$

with Δ_* given in Eq. (4). [Here, we have switched to spherical coordinates again.] If there is a solution of (94) with eigenvalue $\kappa<\lambda$, then one has $W<0$ if the eigenfunction of problem (94) is used as a test function, and this means that the equilibrium is unstable. The lowest eigenvalue is in fact

$$\kappa_0 R=z_1\approx 5.76<\lambda R,\quad j_2(z_1)=0 \qquad (95)$$

and is obtained for

$$X=S_3(r)T_3(\cos\theta)\ , \qquad (96)$$

where S_3 is explained in Eq. (15) and T_3 in the Appendix. This proves the instability for all equilibria with $m\geq 2$. For the case where there is only one magnetic axis ($m=1$), this mode is marginally stable. In order to prove stability of the lowest equilibrium at least for all axisymmetric modes, the modes that are symmetric with respect to the equatorial plane have to be considered, too. For these, the quantities f and g contribute to W_V and their stabilizing influence has to be estimated. This is a delicate issue and we do not dwell on it further.

Next, the so-called ballooning modes ($n\to\infty$) are considered. Ballooning modes are modes that are strongly localized on closed field lines. They test the equilibrium against perturbations that "balloon out" in regions of unfavorable curvature, i.e., regions where the pressure increases in the direction of the curvature vector of the field line. These modes are driven by the pressure gradient and stabilized by the energy, which is necessary to bend the (in general, sheared) field lines. By choosing a suitable test function in the energy functional, the problem can again be reduced to the solution of an eigenvalue problem for a single scalar amplitude Y [20,17]:

$$\mathbf{B}\cdot\nabla\left[\frac{k^2}{B^2}\mathbf{B}\cdot\nabla Y\right]+2\frac{\mu_0}{B^4}(\mathbf{B}\times\mathbf{k}\cdot\nabla p)(\mathbf{B}\times\mathbf{k}\cdot\mathbf{K})Y+\kappa Y=0\ . \qquad (97)$$

If a field line parameter s is introduced, Eq. (97) is obviously an ordinary differential equation in s. The first term represents the always stabilizing contribution due to field line bending, whereas the coefficient of the second term may change its sign, depending on the curvature being favorable or unfavorable. $\mathbf{K}$ denotes the curvature vector

$$\mathbf{K}=\left[\frac{\mathbf{B}}{B}\cdot\nabla\right]\frac{\mathbf{B}}{B} \qquad (98)$$

of the field line and $\mathbf{k}$ is a vector perpendicular to $\mathbf{B}$, which depends on the chosen test function. This choice has to be made carefully in order to satisfy simultaneously the periodicity constraints and the requirements of localization in a sheared equilibrium (see, for instance [17, p. 397]). As a consequence, the amplitude Y is a so-called quasimode, which is defined for all s, $-\infty\leq s\leq\infty$, and has to satisfy the normalization condition

$$\int_{-\infty}^{+\infty}|Y|^2 ds<\infty\ . \qquad (99)$$

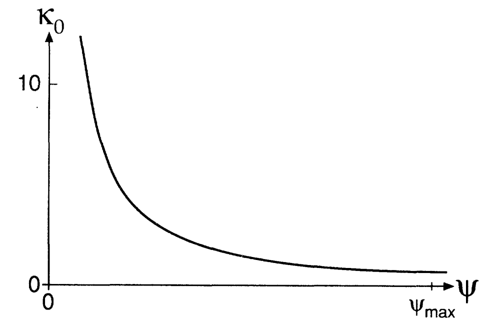

FIG. 6. The lowest ballooning eigenvalue κ_0 vs flux function ψ (both dimensionless).

The equilibrium is stable with respect to ballooning modes if all eigenvalues κ of the problem (97) and (99) for all closed field lines are positive. Owing to axisymmetry, the eigenvalues do not depend on the field line at the same magnetic surface, but may be different for field lines on different surfaces, i.e., κ is actually a function of the surface label ψ. For the lowest equilibrium ($m=1$), the lowest eigenvalue κ_0 was computed numerically using a multiple shooting method. The result, the function $\kappa_0(\psi)$, is shown in Fig. 6. Since κ_0 is everywhere positive, the ballooning modes are stable. This implies that so-called Mercier modes [21] [corresponding to the continuous spectrum of Eq. (97)] are also stable.

Finally, it should be mentioned that the equilibrium is marginally stable with respect to "tilting" modes, which make the spheromak unstable [22]. In fact, the fire ball is marginally stable with respect to an arbitrary rigid motion. The simple reason for this is the fact that a vacuum field with respect to which the fire ball could be oriented is missing.

All these results indicate that the lowest equilibrium could be completely stable. A rigorous proof, however, has yet to be done.

VI. CONCLUSION

One key feature of ball lightning is the remarkable stability in shape—mostly spherical but, to a lesser extent, also ellipsoidal or more complex—lasting seconds or even minutes, which is difficult to explain without either a fluid dynamic or plasma dynamic model. The fact that the large majority of ball lightning observations are made during thunderstorms favors a plasma dynamic explanation. Once the framework of MHD is accepted, it is argued in this paper that MHD equilibria appropriate for describing fireballs should be magnetic field-free outside the plasma region. So far only one family of such fireball equilibria, all with spherical boundary but with different numbers m of magnetic axes, has been known. All but the lowest of these equilibria turn out to be MHD unstable. The lowest one has been examined in greater detail: it is marginally stable with respect to modes, which are axisymmetric ($n=0$, where n is the toroidal wave number) and antisymmetric with respect to the equatorial plane, and stable with respect to ballooning modes ($n\rightarrow\infty$). However, the question of complete stability has not yet been answered. Characteristic quantities have been computed and typical values of magnetic field and current are shown to be in agreement with expectations from ordinary lightning. Finally, perturbations of the linear profiles along with deformations of the spherical boundary are shown to be possible; this makes it likely that fireball equilibria with nonspherical boundary (in accordance with observations) exist.

Another key feature of ball lightning, besides stability in shape, is temporal stability. To be honest, it should be stated that a big problem with all plasma models (and other models as well) is that the fireball, owing to thermal loss, should cool down in milliseconds, not seconds, if there is no internal or external energy source available [2 (p. 133),8], and the model discussed here makes no exception in this respect. To make things worse, no energy source has been identified so far. Moreover, ball lightning observations in closed rooms and even in an airplane [23] virtually rule out external sources such as high-frequency waves and also make the initial storage of combustible substances in the fireball difficult to explain. So, the longevity of the fireball is still an enigma.

ACKNOWLEDGMENTS

One of the authors (D.L.) gratefully acknowledges a discussion with Jin Li, who pointed out that the ordinary virial theorem does not exclude equilibria with spatially decaying magnetic field if the pressure is nonzero outside the plasma. He also gratefully acknowledges valuable discussions with Rita Mayer-Spasche and Günther Spies. The other author (R.K.) would like to thank the Deutsche Forschungsgemeinschaft (DFG) for financial support. This work was performed under the terms of the agreement on association between Max-Planck-Institut für Plasmaphysik and Euratom.

APPENDIX

In this Appendix some properties of the solutions of Eq. (13) are listed for the reader's convenience, especially completeness and orthogonality of the T_n, which are important for the uniqueness of the fireball solution in a spherical domain.

The ansatz $T(t)=\sqrt{1-t^2}P(t)$ transforms Eq. (13) into

$$(1-t^2)\frac{d^2P}{dt^2}-2t\frac{dP}{dt}+\left[c-\frac{1}{1-t^2}\right]P=0 . \qquad \text{(A1)}$$

Solutions of Eq. (A1), which are regular on the interval $-1\leq t\leq 1$, are the associated Legendre functions of first order, P_n^1, $n\in\mathbb{N}$ [15]. The corresponding eigenvalues are $c=c_n=n(n+1)$. The T_n can, therefore, be expressed by P_n^1 or by ordinary Legendre polynomials P_n:

$$T_n(t)=\sqrt{1-t^2}P_{n-1}^1$$
$$=(1-t^2)\frac{d}{dt}P_{n-1}=(n-1)(P_{n-2}-tP_{n-1}) . \qquad \text{(A2)}$$

The first four functions read

$$T_2=1-t^2,\quad T_3=3(1-t^2)t ,$$
$$T_4=\tfrac{3}{2}(1-t^2)(5t^2-1),\quad T_5=\tfrac{5}{2}(1-t^2)(7t^3-3t) . \qquad \text{(A3)}$$

It is, furthermore, reasonable to introduce the scalar product

$$(f,g):=\int_{-1}^{+1}f^*(t)g(t)\frac{dt}{1-t^2} , \qquad \text{(A4)}$$

with respect to which Eq. (13) is Hermitean and the eigenfunctions T_n are orthogonal. The corresponding orthogonality relation can easily be deduced from the analogous one for ordinary Legendre polynomials and reads

$$\int_{-1}^{+1}T_n(t)T_{n'}(t)\frac{dt}{1-t^2}=\frac{2n(n-1)}{2n-1}\delta_{nn'} . \qquad \text{(A5)}$$

Moreover, the T_n are complete on the interval $-1\leq t\leq 1$, i.e., every function $f(t)$ with $(f,f)<\infty$ can be expanded in a series of T_n, $n\geq 2$. This is expressed in the relation

$$\sum_{n=2}^{\infty}\frac{2n-1}{2n(n-1)}T_n(t)T_n(t')=(1-t^2)\delta(t-t') , \qquad \text{(A6)}$$

which can likewise be derived from its equivalent for Legendre polynomials.

[1] D. Finkelstein and J. Rubinstein, Phys. Rev. **135**, A390 (1964).
[2] S. Singer, *The Nature of Ball Lightning* (Plenum, New York, 1971).
[3] J. D. Barry, *Ball Lightning and Bead Lightning* (Plenum, New York, 1980).
[4] B. U. Smirnov, Phys. Rep. **152**, 177 (1987).
[5] K. L. E. Nickel, in *Science of Ball Lightning (Fire Ball)*, edited by Yoshi-Hiko Ohtsuki (World Scientific, Singapore, 1988).
[6] J. Hill, Philos. Trans. R. Soc. London, Ser. A **185**, 213 (1894).
[7] R. Lüst and A. Schlüter, Z. Astrophys. **34**, 263 (1954).
[8] P. L. Kapitza, Dokl. Akad. Nauk SSSR **101**, 245 (1955).
[9] R. Lüst and A. Schlüter, Z. Naturforsch. Teil A **12**, 850 (1957); H. Grad and H. Rubin (unpublished); V. D. Shafranov, Sov. Phys. JETP **6**, 545 (1958).
[10] K. Prendergast, Astrophys. J. **123**, 498 (1956).
[11] G. K. Morikawa, Phys. Fluids **12**, 1648 (1969); G. K. Morikawa and E. Rebhan, *ibid.* **13**, 497 (1970); T. Yeh and G. K. Morikawa, *ibid.* **14**, 781 (1971).
[12] H. Wu and Y. Chen, Phys. Fluids B **1**, 1753 (1989).
[13] H. Grad (unpublished); Int. J. Fusion Energy **3**, 33 (1985).
[14] A. I. Morozow and L. S. Solov'ev, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1966), Vol. 2.

[15] *Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).

[16] J. Serrin, Arch. Ration. Mech. Anal. **43**, 304 (1971).

[17] J. P. Freidberg, *Ideal Magnetohydrodynamics* (Plenum, New York, 1987).

[18] M. Rosenbluth and M. Bussac, Nucl. Fusion **19**, 489 (1979).

[19] D. Lortz, Z. Naturforsch. Teil A **43**, 1009 (1988).

[20] D. Dobrott, D. Nelson, J. Greene, A. Glasser, M. Chance, and E. Frieman, Phys. Rev. Lett. **39**, 943 (1977), D. Correa-Restrepo, Z. Naturforsch. Teil A **33**, 789 (1978).

[21] C. Mercier, Nucl. Fusion **1**, 47 (1960).

[22] D. Lortz and G. Spies, Phys. Plasmas **1**, 682 (1994).

[23] J. D. Barry and S. Singer, in *Science of Ball Lightning (Fire Ball),* edited by Yoshi-Hiko Ohtsuki (World Scientific, Singapore, 1988).